for $p = \infty$, as

$$\|x\|_{\ell_\infty^N} := \|x\|_\infty = \max_{j=1,\dots,N} |x_j|. \tag{B.72}$$

We use the shorthand ℓ_p to denote the case where $N = \infty$. In this case, it is understood that ℓ_p contains all sequences with finite p-norm. For $N = \infty$, the max in (B.72) is replaced by a sup.

Often the above notations are also used in the case where $0 < p < 1$. In that case, however, we are no longer dealing with norms.

Suppose $\mathcal{F}$ is a class of functions $f : \mathcal{X} \to \mathbb{R}$. The ℓ_∞^N norm of $f \in \mathcal{F}$ *with respect to a sample* $X = (x_1, \dots, x_m)$ is defined as

$$\|f\|_{\ell_\infty^X} := \max_{i=1,\dots,m} |f(x_i)|. \tag{B.73}$$

Likewise,

$$\|f\|_{\ell_p^X} = \|(f(x_1), \dots, f(x_m))\|_{\ell_p^m}. \tag{B.74}$$

L_p Spaces

Given some set $\mathcal{X}$ with a σ-algebra, a measure μ on $\mathcal{X}$, some p in the range $1 \le p < \infty$, and a function $f : \mathcal{X} \to \mathbb{R}$, we define

$$\|f\|_{L_p(\mathcal{X})} := \|f\|_p := \left(\int |f(x)|^p d\mu(x) \right)^{1/p} \tag{B.75}$$

if the integral exists, and

$$\|f\|_{L_\infty(\mathcal{X})} := \|f\|_\infty := \operatorname*{ess\,sup}_{x \in \mathcal{X}} |f(x)|. \tag{B.76}$$

Here, ess sup denotes the essential supremum; that is, the smallest number that upper bounds $|f(x)|$ almost everywhere.

For $1 \le p \le \infty$, we define

$$L_p(\mathcal{X}) := \{ f : \mathcal{X} \to \mathbb{R} \mid \|f\|_{L_p(\mathcal{X})} < \infty \}. \tag{B.77}$$

Here, we have glossed over some details: in fact, these spaces do not consist of functions, but of equivalence classes of functions differing on sets of measure zero. An interesting exception to this rule are reproducing kernel Hilbert spaces (Section 2.2.3). For these, we know that point evaluation of all functions in the space is well-defined: it is determined by the reproducing kernel, see (2.29).

Let $\mathfrak{L}(E, G)$ be the set of all bounded linear operators T between the normed spaces $(E, \|\cdot\|_E)$ and $(G, \|\cdot\|_G)$; in other words, operators such that the image of the (closed) unit ball,

$$U_E := \{ x \in E \mid \|x\|_E \le 1 \}, \tag{B.78}$$

is bounded. The smallest such bound is called the *operator norm*,

$$\|T\| := \sup_{x \in U_E} \|Tx\|_G. \tag{B.79}$$