minimum of

$$R(\mathbf{v}) := \frac{\langle \mathbf{v}, A\mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}. \tag{B.57}$$

The minimizer of R is an eigenvector with eigenvalue $\lambda_{\min}$. Likewise, the largest eigenvalue and its corresponding eigenvector can be found by maximizing R.
Functions $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$, can be defined on symmetric matrices A with eigenvalues in I. To this end, we diagonalize A and apply f to all diagonal elements (the eigenvalues).
Since a symmetric matrix is positive definite if and only if all its eigenvalues are nonnegative, we may choose $f(x) = \sqrt{x}$ to obtain the unique *square root* $\sqrt{A}$ of a positive definite matrix A.

Many statements about matrices generalize in some form to operators on spaces of arbitrary dimension; for instance, Mercer's theorem (Theorem 2.10) can be viewed as a generalized version of a matrix diagonalization, with eigenvectors (or eigenfunctions) ψ_j satisfying $\int_{\mathcal{X}} k(x, x')\psi_j(x')\, d\mu(x') = \lambda_j \psi_j(x)$.

B.3 Functional Analysis

Functional analysis combines concepts from linear algebra and analysis. Consequently, it is also concerned with questions of convergence and continuity. For a detailed treatment, cf. [429, 306, 112].

Cauchy Sequence

Definition B.10 (Cauchy Sequence) *A sequence* $(\mathbf{x}_i)_i := (\mathbf{x}_i)_{i\in\mathbb{N}} = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$ *in a normed space* $\mathcal{H}$ *is said to be a* Cauchy sequence *if for every* $\epsilon > 0$*, there exists an* $n \in \mathbb{N}$ *such that for all* $n', n'' > n$*, we have* $\|\mathbf{x}_{n'} - \mathbf{x}_{n''}\| < \epsilon$*.*

A Cauchy sequence is said to converge *to a point* $\mathbf{x} \in \mathcal{H}$ *if* $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ *as* $n \to \infty$*.*

Banach / Hilbert Space

Definition B.11 (Completeness, Banach Space, Hilbert Space) *A space* $\mathcal{H}$ *is called* complete *if all Cauchy sequences in the space converge.*

A Banach space *is a complete normed space; a* Hilbert space *is a complete dot product space.*

The simplest example of a Hilbert space (and thus also of a Banach space) is again $\mathbb{R}^N$. More interesting Hilbert spaces, however, have *infinite* dimensionality. A number of surprising things can happen in this case. To prevent the nasty ones, we generally assume that the Hilbert spaces we deal with are *separable*,[13] which means that there exists a countable dense subset. A *dense subset* is a set S such that each element of $\mathcal{H}$ is the limit of a sequence in S. Equivalently, the completion of

13. One of the positive side effects of this is that we essentially only have to deal with one Hilbert space: all separable infinite-dimensional Hilbert spaces are equivalent, in a sense that we won't define presently.