In order to compute the dual of (6.72), we have to eliminate x from (6.73) and write it as a function of α. We obtain

$$L(x,\alpha) = -\frac{1}{2}x^\top Kx + \alpha^\top d \tag{6.78}$$

$$= -\frac{1}{2}\alpha^\top AK^{-1}A^\top\alpha + \left[d - c^\top K^{-1}A^\top\right]\alpha - \frac{1}{2}c^\top K^{-1}c. \tag{6.79}$$

In (6.78) we used (6.74) and (6.76) directly, whereas in order to eliminate x completely in (6.79) we solved (6.74) for $x = -K^{-1}(c + A^\top\alpha)$. Ignoring constant terms this leads to the dual quadratic optimization problem,

Dual Quadratic Program

$$\begin{aligned} \underset{\alpha}{\text{maximize}} \quad & -\frac{1}{2}\alpha^\top AK^{-1}A^\top\alpha + \left[d - c^\top K^{-1}A^\top\right]\alpha, \\ \text{subject to} \quad & \alpha \geq 0. \end{aligned} \tag{6.80}$$

The surprising fact about the dual problem (6.80) is that the constraints become significantly simpler than in the primal (6.72). Furthermore, if $n < m$, we also obtain a more compact representation of the quadratic term.

There is one aspect in which (6.80) differs from its linear counterpart (6.70): if we dualize (6.80) again, we do not recover (6.72) but rather a problem very similar in structure to (6.80). Dualizing (6.80) twice, however, we recover the dual itself (Problem 6.13 deals with this matter in more detail).

6.4 Interior Point Methods

Let us now have a look at simple, yet efficient optimization algorithms for constrained problems: interior point methods.

An interior point is a pair of variables (x, α) that satisfies both primal and dual constraints. As already mentioned before, finding a set of vectors $(\bar{x}, \bar{\alpha})$ that satisfy the KKT conditions is sufficient to obtain a solution in $\bar{x}$. Hence, all we have to do is devise an algorithm which solves (6.74)–(6.77), for instance, if we want to solve a quadratic program. We will focus on the quadratic case — the changes required for linear programs merely involve the removal of some variables, simplifying the equations. See Problem 6.14 and [555, 512] for details.

6.4.1 Sufficient Conditions for a Solution

We need a slight modification of (6.74)–(6.77) in order to achieve our goal: rather than the inequality (6.75), we are better off with an equality and a positivity constraint for an additional variable, i.e. we transform $Ax + d \leq 0$ into $Ax + d + \xi =$