Additionally, by setting one of the α_i to 0, we see that $\bar{\alpha}_i c_i(\bar{x}) \geq 0$. The only way to satisfy this is by having

$$\bar{\alpha}_i c_i(\bar{x}) = 0 \text{ for all } i \in [n]. \tag{6.42}$$

Eq. (6.42) is often referred to as the KKT condition [283, 312]. Finally, combining (6.42) and $c_i(x) \leq 0$ with the second inequality in (6.40) yields $f(\bar{x}) \leq f(x)$ for all feasible x. This proves that $\bar{x}$ is optimal. ■

We can immediately extend Theorem 6.21 to accommodate equality constraints by splitting them into the conditions $e_i(x) \leq 0$ and $e_i(x) \geq 0$. We obtain:

Theorem 6.22 (Equality Constraints) *Assume an optimization problem of the form (6.38), where $f, c_i, e_j : \mathbb{R}^m \to \mathbb{R}$ for $i \in [n]$ and $j \in [n']$ are arbitrary functions, and a Lagrangian*

$$L(x, \alpha, \beta) := f(x) + \sum_{i=1}^{n} \alpha_i c_i(x) + \sum_{j=1}^{n'} \beta_j e_j(x) \text{ where } \alpha_i \geq 0 \text{ and } \beta_j \in \mathbb{R}. \tag{6.43}$$

If a set of variables $(\bar{x}, \bar{\alpha}, \bar{\beta})$ with $\bar{x} \in \mathbb{R}^m$, $\bar{\alpha} \in [0, \infty)$, and $\bar{\beta} \in \mathbb{R}^{n'}$ exists such that for all $x \in \mathbb{R}^m$, $\alpha \in [0, \infty)^n$, and $\beta \in \mathbb{R}^{n'}$,

$$L(\bar{x}, \alpha, \beta) \leq L(\bar{x}, \bar{\alpha}, \bar{\beta}) \leq L(x, \bar{\alpha}, \bar{\beta}), \tag{6.44}$$

then $\bar{x}$ is a solution to (6.38).

Now we determine when the conditions of Theorem 6.21 are necessary. We will see that convexity and sufficiently "nice" constraints are needed for (6.40) to become a necessary condition. The following lemma (see [345]) describes three *constraint qualifications*, which will turn out to be exactly what we need.

Feasible Region

Lemma 6.23 (Constraint Qualifications) *Denote by $\mathcal{X} \subset \mathbb{R}^m$ a convex set, and by $c_1, \ldots, c_n : \mathcal{X} \to \mathbb{R}$ n convex functions defining a feasible region by*

$$X := \{x | x \in \mathcal{X} \text{ and } c_i(x) \leq 0 \text{ for all } i \in [n]\}. \tag{6.45}$$

Equivalence Between Constraint Qualifications

Then the following additional conditions on c_i are connected by $(i) \Longleftrightarrow (ii)$ and $(iii) \Longrightarrow (i)$.

(i) There exists an $x \in \mathcal{X}$ such that for all $i \in [n]$ $c_i(x) < 0$ (Slater's condition [500]).

(ii) For all nonzero $\alpha \in [0, \infty)^n$ there exists an $x \in \mathcal{X}$ such that $\sum_{i=1}^{n} \alpha_i c_i(x) \leq 0$ (Karlin's condition [281]).

(iii) The feasible region X contains at least two distinct elements, and there exists an $x \in X$ such that all c_i are strictly convex at x wrt. X (Strict constraint qualification).

The connection $(i) \Longleftrightarrow (ii)$ is also known as the Generalized Gordan Theorem [164]. The proof can be skipped if necessary. We need an auxiliary lemma which we state without proof (see [345, 435] for details).