Example 4.22 (Inhomogeneous Polynomial Kernels $k(x,x') = (\langle x,x'\rangle + 1)^p$) *Likewise, let us analyze $k(\xi) = (1+\xi)^p$ for $p > 0$. Again, we expand k in a series of Legendre Polynomials, to obtain [209, 7.127]*

$$\int_{-1}^{1} \mathcal{P}_n(\xi)(\xi+1)^p d\xi = \frac{2^{p+1}\Gamma^2(p+1)}{\Gamma(p+2+n)\Gamma(p+1-n)}. \tag{4.72}$$

For $p \in \mathbb{N}$, all terms with $n > p$ vanish, and the remainder is positive. For non-integer p, however, (4.72) may change its sign. This is due to $\Gamma(p+1-n)$. In particular, for any $p \notin \mathbb{N}$ (with $p > 0$), we have $\Gamma(p+1-n) < 0$ for $n = \lceil p \rceil + 1$. This violates condition (4.68), hence such kernels cannot be used in SV machines unless $p \in \mathbb{N}$.

Example 4.23 (Vovk's Real Polynomial $k(x,y) = \frac{1-\langle x,y\rangle^p}{1-(\langle x,y\rangle)}$ *with* $p \in \mathbb{N}$ [460]) *This kernel can be written as $k(\xi) = \sum_{n=0}^{p-1} \xi^n$, hence all the coefficients $a_i = 1$, which means that the kernel can be used regardless of the dimensionality of the input space.*
Likewise we can analyze an infinite power series.

Example 4.24 (Vovk's Infinite Polynomial $k(x,x') = (1-(\langle x,x'\rangle))^{-1}$ [460]) *This kernel can be written as $k(\xi) = \sum_{n=0}^{\infty} \xi^n$, hence all the coefficients $a_i = 1$. The flat spectrum of the kernel suggests poor generalization properties.*

Example 4.25 (Neural Network Kernels $k(x,x') = \tanh(a + \langle x,x'\rangle)$) *We next show that $k(\xi) = \tanh(a+\xi)$ is never positive definite, no matter how we choose the parameters.*

The technique is identical to that of Examples 4.21 and 4.22: we have to show that the kernel does not satisfy the conditions of Theorem 4.18. Since this is very technical (and is best done using computer algebra programs such as Maple), we refer the reader to [401] for details, and explain how the method works in the simpler case of Theorem 4.19. Expanding $\tanh(a+\xi)$ into a Taylor series yields

$$\tanh a + \xi \tfrac{1}{\cosh^2 a} - \xi^2 \tfrac{\tanh a}{\cosh^2 a} - \tfrac{\xi^3}{3}(1-\tanh^2 a)(1-3\tanh^2 a) + O(\xi^4). \tag{4.73}$$

We now analyze (4.73) coefficient-wise. Since the coefficients have to be nonnegative, we obtain $a \in [0,\infty)$ from the first term, $a \in (-\infty, 0]$ from the third term, and $|a| \in [\text{arctanh}\,\frac{1}{3}, \text{arctanh}\,1]$ from the fourth term . This leaves us with $a \in \emptyset$, hence there are no parameters for which this kernel is positive definite.

4.7 Multi-Output Regularization

So far in this chapter we only considered scalar functions $f : \mathcal{X} \to \mathcal{Y}$. Below we will show that under rather mild assumptions on the symmetry properties of $\mathcal{Y}$, there exist no other vector valued extensions to $\Upsilon^*\Upsilon$ than the trivial extension, i.e., the application of a scalar regularization operator to each of the dimensions of $\mathcal{Y}$ separately. The reader not familiar with group theory may want to skip the more detailed discussion given below.