lor series expansion has only nonnegative coefficients;

$$k(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \text{ with } a_n \geq 0. \tag{4.69}$$

Therefore, all we have to do in order to check whether a particular kernel may satisfy Mercer's condition, is to look at its polynomial series expansion, and check the coefficients.

We note that (4.69) is a more stringent condition than (4.68). In other words, in order to prove positive definiteness for arbitrary dimensions it suffices to show that the Taylor expansion contains only positive coefficients. On the other hand, in order to prove that a candidate for a kernel function will never be positive definite, it is sufficient to show this for (4.68) where $\mathcal{P}_n^N = \mathcal{P}_n$, i.e. for the Legendre Polynomials.

Eigenvector Decomposition We conclude this section with an explicit representation of the eigensystem of $k(\langle x, x' \rangle)$. For a proof see [511].

Lemma 4.20 (Eigenvector Decomposition of Dot Product Kernels) *Denote by $k(\langle x, x' \rangle)$ a kernel on $S_{N-1} \times S_{N-1}$ satisfying condition (4.68) of Theorem 4.18. Then the eigenvectors of k are given by*

$$\psi_{n,j} = Y_{n,j}^N, \text{ with eigenvalues } \lambda_{n,j} = a_n \frac{|S_{N-1}|}{M(N,n)} \text{ of multiplicity } M(N,n). \tag{4.70}$$

In other words, $\frac{a_n}{M(N,n)}$ determines the regularization properties of $k(\langle x, x' \rangle)$.

4.6.2 Examples and Applications

In the following we will analyze a few kernels, and state under which conditions they may be used as SV kernels.

Example 4.21 (Homogeneous Polynomial Kernels $k(x, x') = \langle x, x' \rangle^p$) *As we showed in Chapter 2, this kernel is positive definite for $p \in \mathbb{N}$. We will now show that for $p \notin \mathbb{N}$ this is never the case.*

We thus have to show that (4.68) cannot hold for an expansion in terms of Legendre Polynomials ($N = 3$). From [209, 7.126.1], we obtain for $k(\xi) = |\xi|^p$ (we need $|\xi|$ to make k well-defined),

$$\int_{-1}^{1} \mathcal{P}_n(\xi) |\xi|^p d\xi = \frac{\sqrt{\pi}\,\Gamma(p+1)}{2^p \Gamma\left(1 + \frac{p}{2} - \frac{n}{2}\right) \Gamma\left(\frac{3}{2} + \frac{p}{2} + \frac{n}{2}\right)} \text{ if } n \text{ even.} \tag{4.71}$$

For odd n, the integral vanishes, since $\mathcal{P}_n(-\xi) = (-1)^n \mathcal{P}_n(\xi)$. In order to satisfy (4.68), the integral has to be nonnegative for all n. One can see that $\Gamma\left(1 + \frac{p}{2} - \frac{n}{2}\right)$ is the only term in (4.71) that may change its sign. Since the sign of the Γ function alternates with period 1 for $x < 0$ (and has poles for negative integer arguments), we cannot find any p for which $n = 2\lfloor \frac{p}{2} + 1 \rfloor$ and $n = 2\lceil \frac{p}{2} + 1 \rceil$ correspond to positive values of the integral.