that a valid feature map for this kernel can be defined coordinate-wise as

$$\Phi_{\mathbf{m}}(\mathbf{x}) = \sqrt{\frac{d!}{\prod_{i=1}^{n}[\mathbf{m}]_i!}} \prod_{i=1}^{n} [\mathbf{x}]_i^{[\mathbf{m}]_i} \tag{2.95}$$

for every $\mathbf{m} \in \mathbb{N}^n$, $\sum_{i=1}^{n}[\mathbf{m}]_i = d$ *(i.e., every such* $\mathbf{m}$ *corresponds to one dimension of* $\mathcal{H}$*).*

2.3 (Inhomogeneous Polynomial Kernel ••) *Prove that the kernel (2.70) induces a feature map into the space of all monomials up to degree d. Discuss the role of c.*

2.4 (Eigenvalue Criterion of Positive Definiteness •) *Prove that a symmetric matrix is positive definite if and only if all its eigenvalues are nonnegative (see Appendix B).*

2.5 (Dot Products are Kernels •) *Prove that dot products (Definition B.7) are positive definite kernels.*

2.6 (Kernels on Finite Domains ••) *Prove that for finite* $\mathcal{X}$*, say* $\mathcal{X} = \{x_1, \ldots, x_m\}$*, k is a kernel if and only if the* $m \times m$ *matrix* $(k(x_i, x_j))_{ij}$ *is positive definite.*

2.7 (Positivity on the Diagonal •) *From Definition 2.5, prove that a kernel satisfies* $k(x, x) \geq 0$ *for all* $x \in \mathcal{X}$*.*

2.8 (Cauchy-Schwarz for Kernels ••) *Give an elementary proof of Proposition 2.7.*

Hint: start with the general form of a symmetric 2×2 *matrix, and derive conditions for its coefficients that ensure that it is positive definite.*

2.9 (PD Kernels Vanishing on the Diagonal •) *Use Proposition 2.8 to prove that a kernel satisfying* $k(x, x) =$ *for all* $x \in \mathcal{X}$ *is identically zero.*

How does the RKHS look in this case? Hint: use (2.31).

2.10 (Two Kinds of Positivity •) *Give an example of a kernel which is positive definite according to Definition 2.5, but not positive in the sense that* $k(x, x') \geq 0$ *for all* x, x'*.*

Give an example of a kernel where the contrary is the case.

2.11 (General Coordinate Transformations •) *Prove that if* $\sigma : \mathcal{X} \to \mathcal{X}$ *is a function, and* $k(x, x')$ *is a kernel, then* $k(\sigma(x), \sigma(x'))$ *is a kernel, too.*

2.12 (Positivity on the Diagonal •) *Prove that positive definite kernels are positive on the diagonal,* $k(x, x) \geq 0$ *for all* $x \in \mathcal{X}$*. Hint: use* $m = 1$ *in (2.15).*

2.13 (Symmetry of Complex Kernels ••) *Prove that complex-valued positive definite kernels are symmetric (2.18).*

2.14 (Real Kernels vs. Complex Kernels •) *Prove that a real matrix satisfies (2.15) for all* $c_i \in \mathbb{C}$ *if and only if it is symmetric and it satisfies (2.15) for* real *coefficients* c_i*.*

Hint: decompose each c_i *in (2.15) into real and imaginary parts.*