RKHS

In view of the properties (2.29) and (2.30), this space is usually called a *reproducing kernel Hilbert space (RKHS)*.

In general, an RKHS can be defined as follows.

Definition 2.9 (Reproducing Kernel Hilbert Space) *Let $\mathcal{X}$ be a nonempty set (often called the index set) and $\mathcal{H}$ a Hilbert space of functions $f : \mathcal{X} \to \mathbb{R}$. Then $\mathcal{H}$ is called a reproducing kernel Hilbert space endowed with the dot product $\langle \cdot, \cdot \rangle$ (and the norm $\|f\| := \sqrt{\langle f, f \rangle}$) if there exists a function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ with the following properties.*

Reproducing Property

1. *k has the reproducing property*[3]

$$\langle f, k(x, \cdot) \rangle = f(x) \text{ for all } f \in \mathcal{H}; \tag{2.34}$$

in particular,

$$\langle k(x, \cdot), k(x', \cdot) \rangle = k(x, x'). \tag{2.35}$$

Closed Space

2. *k spans $\mathcal{H}$, i.e. $\mathcal{H} = \overline{\text{span}\,\{k(x, \cdot) | x \in \mathcal{X}\}}$ where $\overline{X}$ denotes the completion of the set X (cf. Appendix B).*

On a more abstract level, an RKHS can be defined as a Hilbert space of functions f on $\mathcal{X}$ such that all evaluation functionals (the maps $f \mapsto f(x')$, where $x' \in \mathcal{X}$) are continuous. In that case, by the Riesz representation theorem (e.g., [429]), for each $x' \in \mathcal{X}$ there exists a unique function of x, called $k(x, x')$, such that

$$f(x') = \langle f, k(., x') \rangle . \tag{2.36}$$

It follows directly from (2.35) that $k(x, x')$ is symmetric in its arguments (see Problem 2.28) and satisfies the conditions for positive definiteness.

Uniqueness of k

Note that the RKHS uniquely determines k. This can be shown by contradiction: assume that there exist two kernels, say k and k', spanning the same RKHS $\mathcal{H}$. From Problem 2.28 we know that both k and k' must be symmetric. Moreover, from (2.34) we conclude that

$$\langle k(x, \cdot), k'(x', \cdot) \rangle_{\mathcal{H}} = k(x, x') = k'(x', x). \tag{2.37}$$

In the second equality we used the symmetry of the dot product. Finally, symmetry in the arguments of k yields $k(x, x') = k'(x, x')$ which proves our claim.

2.2.4 The Mercer Kernel Map

Section 2.2.2 has shown that any positive definite kernel can be represented as a dot product in a linear space. This was done by explicitly constructing a (Hilbert) space that does the job. The present section will construct another Hilbert space.

3. Note that this implies that each $f \in \mathcal{H}$ is actually a single function whose values at any $x \in \mathcal{X}$ are well-defined. In contrast, L_2 Hilbert spaces usually do not have this property. The elements of these spaces are equivalence classes of functions that disagree only on sets of measure 0; cf. footnote 15 in Section B.3.